Journal of Global Optimization **17:** 185–200, 2000.

Basis Reduction for the Shakedown Problem for Bounded Kinematic Hardening Material

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(Received for publication 18 August 2000)

Abstract. Limit and shakedown analysis are effective methods for assessing the load carrying capacity of a given structure. The elasto–plastic behavior of the structure subjected to loads varying in a given load domain is characterized by the shakedown load factor, defined as the maximum factor which satisfies the sufficient conditions stated in the corresponding static shakedown theorem. The finite element dicretization of the problem may lead to very large convex optimization. For the effective solution a basis reduction method has been developed that makes use of the special problem structure for perfectly plastic material. The paper proposes a modified basis reduction method for direct application to the two-surface plasticity model of bounded kinematic hardening material. The considered numerical examples show an enlargement of the load carrying capacity due to bounded hardening.

Key words: Basis reduction; Convex optimization; FEM; Shakedown analysis

1. Introduction

The load carrying capacity is a central question in the design and the analysis of engineering structures made of ductile material. As the elastic structural response gives only a fictitious safety margin and a fictitious reliability estimate, it is neither practical nor economic to estimate the load carrying capacity by considering only those loadings with a purely elastic structural response. On the other hand, even if it is possible to compute the inelastic structural response for a given load history in a time stepping incremental analysis, the past loading may not have been recorded and the future loading cannot be foreseen. Additionally, the structural reliability must be measured against all possible loadings, which are not a finite number and the different parameters of the constitutive equations cannot be precisely measured. All these arguments recommend the direct methods of plasticity, i.e. limit and shakedown analysis, methods which are effective from the numerical point of view and useful in the study of complex structures.

The static approach to limit and shakedown analysis poses a convex optimization problem with an infinite number of constraints. This approach is based on a mathematical formulation of the conditions which the structure must satisfy such that critical states corresponding to plastic collapse, incremental collapse (ratchetting) or alternating plasticity (Low Cycle Fatigue, LCF) are not attained. In fact, these conditions assure the boundedness for all possible loading histories of the total plastically dissipated plastic energy of the structure. The structure must be in equilibrium and the constitutive equations must be satisfied in any material point. The static limit and shakedown theorem has been formulated by Melan for perfectly plastic and for unbounded kinematic hardening material (Melan, 1938). Using a two–surface plasticity model, a generalization to bounded kinematic hardening has been proposed in (Weichert and Gross-Weege, 1988). It turns out that only few information are relevant, e.g. for monotone loading limit analysis shows that no elastic data enters the problem. Similarly, for certain cyclic or general time invariant load histories shakedown analysis needs few characteristic data of the hardening behavior (Zhang, 1991), (Stein et al., 1993).

For a Finite Element (FE) discretisation a finite but generally large number of constraints is achieved. The basis reduction method keeps only a small number of unknowns. It was developed for linear optimization in (Shen, 1986) making use of the special structure of the shakedown problem for perfectly plastic material. In the same constitutive setting the method has been extended to nonlinear optimization in (Gross-Weege, 1996), (Heitzer, 1999), (Staat and Heitzer, 1997), (Zhang, 1991). The extension to the more realistic bounded kinematic hardening material has been achieved in (Zhang, 1991), (Stein et al., 1993) by use of the overlay model (also called fraction or multiple subvolume model) which preserves the characteristic structure of the perfectly plastic formulation. However, before this approach can be used with a commercial FE code it would be necessary to implement the overlay model for different types of finite elements.

It is the purpose of this contribution to propose a modified basis reduction method for the structure of a two-surface plasticity formulation of bounded kinematic hardening. It can be used for any type of finite elements with no need to make any changes in the plasticity section of the FE code. The new method is implemented in the general FE–code PERMAS (PERMAS, 1988). An increase of the load carrying capacity due to hardening is shown in some numerical examples.

2. Bounded kinematic hardening

For describing the theoretical frame we will use the Generalized Standard Material Model (Halphen and Nguyen, 1975).

An elastic–plastic body of finite volume V with a sufficiently smooth surface ∂V, subjected to the quasi–statical thermo–mechanical loads $\mathbf{P}(t)$ varying in the load domain $\mathcal{L}$ is considered. The hypothesis of small displacements and small strains is made and the strains are decomposed in:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^E + \boldsymbol{\varepsilon}^p + \boldsymbol{\varepsilon}^{th} \text{ with } \boldsymbol{\varepsilon}^{th} = \alpha_t\, \theta\, \mathbf{I}.$$

Here α_t is the coefficient of isotropic temperature expansion and $\theta = T - T_0$, where T_0 is a reference temperature.

At each time t the load consists of body forces, surface tractions (acting on ∂V_p), given displacements (on ∂V_u, where $\partial V_p + \partial V_u = \partial V$) or prescribed temperatures (in V).

The observable variables are the total strain $\boldsymbol{\varepsilon}$ and the temperature T. The internal variables $\boldsymbol{\varepsilon}^p$ and κ will describe the influence of the past history. The thermodynamic potential ψ has the form

$$\psi = \psi(\boldsymbol{\varepsilon}^E, \kappa\ , T) = \psi_e(\boldsymbol{\varepsilon}^E, T) + \psi_p(\kappa).$$

It is assumed that $\rho\psi_p$ is a quadratic form in the variable κ and

$$\rho\psi_e = \frac{1}{2}(\boldsymbol{\varepsilon}^E - \alpha_t\theta\mathbf{I}) : \mathbf{E} : (\boldsymbol{\varepsilon}^E - \alpha_t\theta\mathbf{I}) + C_\varepsilon\theta^2,$$

where ρ is the mass density, $\mathbf{E}$ is the elasticity tensor, C_ε is the specific heat at constant strain.

The associated variables, i.e. the observable stresses $\boldsymbol{\sigma}$ and the internal back–stresses $\boldsymbol{\pi}$, are derived from the potential ψ as follows:

$$\boldsymbol{\sigma} = \rho\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}^E}\ ; \boldsymbol{\pi} = \rho\frac{\partial\psi}{\partial\kappa}.$$

The internal variable κ is a kinematic hardening variable and its associated variable $\boldsymbol{\pi}$ is associated with the center of the elastic domain.

Assuming the decoupling between intrinsic (mechanical) dissipation and thermal dissipation, the Clausius–Duhem inequality gives:

$$\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p - \boldsymbol{\pi} : \dot{\kappa} \geqslant 0.$$

The linear kinematic hardening corresponds to the translation of the loading surface:

$$F[\boldsymbol{\sigma} - \boldsymbol{\pi}] = \sigma_y^2.$$

The interior of the loading surface $\{\boldsymbol{\sigma} \mid F[\boldsymbol{\sigma} - \boldsymbol{\pi}] < \sigma_y^2\}$ is the elastic domain which is described by the function F and the yield stress σ_y. The homogeneous von Mises function $F[\boldsymbol{\sigma}] = 3\boldsymbol{\sigma}^D : \boldsymbol{\sigma}^D$ of degree 2 with the deviatoric stress $\boldsymbol{\sigma}^D = \boldsymbol{\sigma} - \frac{1}{3}(\mathrm{tr}\,\boldsymbol{\sigma})\mathbf{I}$ is the simplest smooth function which can be considered for isotropic, plastically incompressible materials.

The stress $\boldsymbol{\sigma}$ is bounded by the ultimate stress σ_u and the limit surface is described with the same von Mises function:

$$F[\boldsymbol{\sigma}] \leqslant \sigma_u^2.$$

The elastic domain remains always in the limit surface and any stress point in it may be reached if and only if

$$F[\boldsymbol{\pi}] \leqslant \left(\sigma_u - \sigma_y\right)^2.$$

The associated normality hypothesis is made for the plastic flow:

$$\dot{\kappa} = \dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\frac{\partial F}{\partial \boldsymbol{\sigma}}[\boldsymbol{\sigma} - \boldsymbol{\pi}], \text{ with } \begin{cases} \dot{\lambda} = 0, \text{ if } F[\boldsymbol{\sigma} - \boldsymbol{\pi}] < \sigma_y^2 \\ \dot{\lambda} = 0, \text{ if } F[\boldsymbol{\sigma} - \boldsymbol{\pi}] = \sigma_y^2 \text{ and} \\ \qquad (\dot{\boldsymbol{\sigma}} - \dot{\boldsymbol{\pi}}) : \dfrac{\partial F}{\partial \boldsymbol{\sigma}}[\boldsymbol{\sigma} - \boldsymbol{\pi}] < 0 \\ \dot{\lambda} > 0, \text{ else.} \end{cases}$$

The stresses $\boldsymbol{\sigma}$ are decomposed into fictitious elastic stresses $\boldsymbol{\sigma}^E$ and residual stresses $\boldsymbol{\rho}$ by

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^E + \boldsymbol{\rho}. \tag{1}$$

$\boldsymbol{\sigma}^E = \mathbf{E} : \boldsymbol{\varepsilon}$ are stresses which would appear in an infinitely elastic material. The residual stresses (eigenstresses) $\boldsymbol{\rho}$ result from plastic deformations and satisfy the homogeneous static equilibrium and boundary conditions

$$\operatorname{div} \boldsymbol{\rho} = \mathbf{0} \quad \text{in } V \tag{2}$$

$$\boldsymbol{\rho}\, \mathbf{n} = \mathbf{0} \quad \text{on } \partial V_p. \tag{3}$$

3. Lower bound approach of shakedown

A body *shakes down elastically* for the given history of loading $\mathbf{P}(t)$ varying in $\mathcal{L}$ if the plastic strains $\boldsymbol{\varepsilon}^p(t)$ become stationary, i.e.

$$\lim_{t\to\infty} \dot{\boldsymbol{\varepsilon}}^p(\mathbf{x}, t) = \mathbf{0}, \quad \forall\, \mathbf{x} \in V. \tag{4}$$

and the total plastic energy dissipation in the structure for the whole load history is bounded, i.e. $W_p = \int_0^\infty \int_V \boldsymbol{\sigma}(\mathbf{x}, t) : \dot{\boldsymbol{\varepsilon}}^p(\mathbf{x}, t) d\mathbf{x}dt < \infty$. Therefore, a body shakes down if independent of the loading history the body approaches asymptotically an elastic limit state.

The extended static theorem of shakedown for a bounded kinematic hardening material can be formulated as follows (Stein et al., 1993):

If there exist a time–independent back–stresses field $\boldsymbol{\pi}(\mathbf{x})$ satisfying

$$F[\boldsymbol{\pi}(\mathbf{x})] \leqslant \left(\sigma_u(\mathbf{x}) - \sigma_y(\mathbf{x})\right)^2,$$

a factor $\alpha > 1$ and a time–independent residual stress field $\boldsymbol{\rho}(\mathbf{x})$ such that

$$F[\alpha\boldsymbol{\sigma}^E(\mathbf{x}, t) + \boldsymbol{\rho}(\mathbf{x}) - \boldsymbol{\pi}(\mathbf{x})] \leqslant \sigma_y^2(\mathbf{x}) \tag{5}$$

holds for all possible loads $\mathbf{P}(t) \in \mathcal{L}$ and for all material points $\mathbf{x}$, then the structure will shake down elastically under the given load domain $\mathcal{L}$.

The greatest value α_{sd} for which the theorem holds is called *shakedown*-factor. This lower bound approach leads to the convex optimization problem

$$\begin{aligned}
\max \quad & \alpha && (6)\\
\text{s.t.} \quad & F[\alpha\boldsymbol{\sigma}^E(\mathbf{x},t) + \boldsymbol{\rho}(\mathbf{x}) - \boldsymbol{\pi}(\mathbf{x})] \leqslant \sigma_y^2(\mathbf{x}) \qquad \forall \mathbf{x} \in V\\
& F[\boldsymbol{\pi}(\mathbf{x})] \leqslant \left(\sigma_u(\mathbf{x}) - \sigma_y(\mathbf{x})\right)^2 \qquad \forall \mathbf{x} \in V\\
& \operatorname{div} \boldsymbol{\rho}(\mathbf{x}) = \mathbf{0} \qquad \forall \mathbf{x} \in V\\
& \boldsymbol{\rho}(\mathbf{x})\, \mathbf{n} = \mathbf{0} \qquad \forall \mathbf{x} \in \partial V_p
\end{aligned}$$

with infinitely many constraints, which can be reduced to a finite problem by FEM discretization (see the following sections). Shakedown analysis gives the largest range in which the loads may safely vary with arbitrary load history. If the load domain $\mathcal{L}$ shrinks to a single load point, limit analysis is obtained as a special case. For the perfectly plastic behavior ($\sigma_u = \sigma_y$), the back–stresses $\boldsymbol{\pi}$ are identical zero due to the second inequality. Melan's original theorem for unbounded kinematic hardening can be also deduced from the previous formulation if $\sigma_u \to \infty$. Then the second inequality is not relevant anymore and the back–stresses $\boldsymbol{\pi}$ are free variables.

The 3–dimensional overlay (microelement) model, known also as Besseling's fraction model (Besseling, 1985), was used in (Stein et al., 1993) for solving numerically the problem (6). In the overlay model an infinite number of microelements denoted by the scalar $\xi \in [0, 1]$ are associated with each material point of the given structure $\mathbf{x} \in V$. In a simple model each layer (characterized by a constant ξ) behaves elastic, perfectly plastic. All layers have the same elasticity tensor, but they have different yield stresses denoted by $k(\xi)$. It is assumed that the internal strength $k(\xi)$ is a monotonously increasing function of ξ. Additionally, for each $\mathbf{x}$,

$$k(\mathbf{x}, 0) = \sigma_y(\mathbf{x}), \quad \int_0^1 k(\mathbf{x}, \xi)\, d\xi = \sigma_u(\mathbf{x}).$$

It was proved that the shakedown load factor, i.e. the solution of the problem (6) depends only on the values $\sigma_u(\mathbf{x})$ and $\sigma_y(\mathbf{x})$, i.e. it does not depend on the function $k(\xi)$.

4. Discretization of the problem

For the FEM the structure V is decomposed in NE finite elements with the Gaussian points $\mathbf{x}_i$, $i = 1, \ldots, NG$. The constraints of the optimization problem (6) are checked only in the Gaussian points.

The following abbreviations* are used in this paper:

NE number of elements of the structure

NF number of degrees of freedom of the structure

NG number of Gaussian points of the structure

NS number of stress components in each Gaussian points

NV number of load vertices

Using the displacement approach as in all commercial FEM codes (Argyris and Mlejnik, 1987), (PERMAS, 1988) the following discretized equilibrium equations for the residual stresses can be derived:

$$\sum_{i=1}^{NG} \mathbf{C}_i \boldsymbol{\rho}_i = \mathbf{0}. \tag{7}$$

The element matrices $\mathbf{C}_i$ are calculated by the nodal point displacements and the boundary conditions of the structure such that $\mathbf{C}_i \in \mathbb{R}^{NF \times NS}$ and $\boldsymbol{\rho}_i \in \mathbb{R}^{NS}$ holds. With the abbreviations $\mathbf{C} = (\mathbf{C}_1, \ldots, \mathbf{C}_{NG})$ and $\boldsymbol{\rho}^T = \left(\boldsymbol{\rho}_1^T, \ldots, \boldsymbol{\rho}_{NG}^T\right)$, equation (7) yields

$$\mathbf{C}\boldsymbol{\rho} = \mathbf{0}. \tag{8}$$

With the fictitious elastic stresses $\boldsymbol{\sigma}_i^E(t) = \boldsymbol{\sigma}^E(\mathbf{x}_i, t)$, the residual stresses $\boldsymbol{\rho}_i = \boldsymbol{\rho}(\mathbf{x}_i)$, the back–stresses $\boldsymbol{\pi}_i = \boldsymbol{\pi}(\mathbf{x}_i)$, the yield stresses $\sigma_{y,i} = \sigma_y(\mathbf{x}_i)$ and the ultimate stresses $\sigma_{u,i} = \sigma_u(\mathbf{x}_i)$, the constraints in the extended static shakedown theorem become:

$$\begin{aligned} &F[\alpha\boldsymbol{\sigma}_i^E(t) + \boldsymbol{\rho}_i - \boldsymbol{\pi}_i] \leqslant \sigma_{y,i}^2\,, \\ &F[\boldsymbol{\pi}_i] \leqslant \left(\sigma_{u,i} - \sigma_{y,i}\right)^2, \qquad \forall i = 1, \ldots, NG. \\ &\mathbf{C}\boldsymbol{\rho} = \mathbf{0}. \end{aligned}$$

All vectors $\boldsymbol{\rho}$ which fulfill equation (8) are in the kernel $\mathcal{B}$ (*residual stress space*) of the linear mapping defined by the matrix $\mathbf{C}$. In our case $\mathbf{C} \in \mathbb{R}^{NF \times (NS \cdot NG)}$ and $\boldsymbol{\rho} \in \mathbb{R}^{NS \cdot NG}$. If rigid body movements are excluded, the matrix $\mathbf{C}$ has the maximum rank and its rank is given by the degrees of freedom NF of structure V. Consequently, $\dim \mathcal{B} = NG \cdot NS - NF$.

The discretized shakedown problem of the lower bound approach is:

$$\begin{aligned} \max \quad & \alpha \\ \text{s.t.} \quad & F[\alpha\boldsymbol{\sigma}_i^E(t) + \boldsymbol{\rho}_i - \boldsymbol{\pi}_i] \leqslant \sigma_{y,i}^2 \\ & F[\boldsymbol{\pi}_i] \leqslant \left(\sigma_{u,i} - \sigma_{y,i}\right)^2 \\ & \text{for } i = 1, \ldots, NG,\ P(t) \in \mathcal{L},\ \boldsymbol{\rho} \in \mathcal{B} \text{ and } \boldsymbol{\pi} \in \mathbb{R}^{NS \cdot NG}. \end{aligned} \tag{9}$$

* It is assumed for simplicity that all elements have the same number of Gaussian points.

This problem has $2{\cdot}NS{\cdot}NG{+}1$ unknowns: $\boldsymbol{\rho}_i$, $\boldsymbol{\pi}_i$ and the load factor α. Because of the time dependence of the fictitious elastic stresses $\boldsymbol{\sigma}_i^E$ the number of constraints is still infinite.

We proceed now with the discretization of the load domain assuming that the load boundary ∂V_p remains constant. For problems with variable load boundary see e.g. (König, 1987), (Shen, 1986) for moving loads on plates or (Kapoor and Johnson, 1994) for structures with contact. Additionally, we suppose that the load domain $\mathcal{L}$ is a convex polyhedron with the vertices $\mathbf{P}(k),\ k = 1, \dots, NV$ (*load vertices*). Consequently, any load $\mathbf{P}(t) \in \mathcal{L}$ is given by a convex combination of the $\mathbf{P}(j),\ j = 1, \dots, NV$.

Let $\boldsymbol{\sigma}_i^E(j)$ be the fictitious elastic stress in the Gaussian point $\mathbf{x}_i$ corresponding to the j -th load vertex. Due to the convexity of F the constraints of (9) must be verified only in the load vertices, therefore the mathematical optimization problem (9) is reduced to the following one:

$$\begin{aligned} \max \quad & \alpha \\ \text{s.t.} \quad & F[\alpha\boldsymbol{\sigma}_i^E(j) + \boldsymbol{\rho}_i - \boldsymbol{\pi}_i] \leqslant \sigma_{y,i}^2 \\ & F[\boldsymbol{\pi}_i] \leqslant \left(\sigma_{u,i} - \sigma_{y,i}\right)^2 \\ & \text{for } i = 1, \dots, NG,\ j = 1, \dots, NV,\ \boldsymbol{\rho} \in \mathcal{B} \text{ and } \boldsymbol{\pi} \in \mathbb{R}^{NS\cdot NG}. \end{aligned} \tag{10}$$

The number of constraints is finite and for structures with NG Gaussian points we have to handle $O(NG)$ unknowns and $O(NG)$ constraints. Compared to the perfectly plastic and the unbounded kinematic hardening models, the problem (10) has almost a double number of unknowns. Also the number of inequalities increases by NG because of the limiting conditions $F[\boldsymbol{\pi}_i] \leqslant (\sigma_{u,i} - \sigma_{y,i})^2$.

The number of Gaussian points becomes huge for industrial structures and no effective solution algorithms for the nonlinear optimization problem (10) are available. A method for handling such large–scale optimization problems, method called *basis reduction technique* or *subspace iteration*), was used in (Shen, 1986), (Zhang, 1991), (Heitzer, 1999).

5. The basis reduction technique for perfect plasticity

The subspace technique for the perfectly plastic behavior was proposed in (Shen, 1986) and then extended in (Zhang, 1991), (Gross-Weege, 1996), (Staat and Heitzer, 1997), (Heitzer, 1999).

For perfectly plastic material it holds $\sigma_u = \sigma_y$. Therefore, the back–stresses $\boldsymbol{\pi} = \mathbf{0}$ and the following maximum problem with an arbitrary stress $\boldsymbol{\sigma}^0$ must be solved:

$$\begin{aligned} \max \quad & \alpha \\ \text{s.t.} \quad & F[\alpha\boldsymbol{\sigma}_i^E(j) + \boldsymbol{\rho}_i + \boldsymbol{\sigma}_i^0] \leqslant \sigma_{y,i}^2 \\ & \text{for } i = 1, \dots, NG,\ j = 1, \dots, NV,\ \boldsymbol{\rho} \in \mathcal{B}. \end{aligned} \tag{11}$$

The stress $\boldsymbol{\sigma}^0$ can be in equilibrium with a dead load, for example with the weight of the considered body.

Instead of searching the whole vector space $\mathcal{B}$ for a solution of this problem, we search in a d–dimensional subspace $\mathcal{B}_d$. Iteratively, a different subspace $\mathcal{B}_d^k$ is chosen in the k -th step of the algorithm for improving the current load factor α^{k-1}. The dimension of the chosen subspaces is rather small compared to the dimension of $\mathcal{B}$, typically $\dim \mathcal{B}_d^k = d \leqslant 6$. The subspaces $\mathcal{B}_d^k$ will be generated by d linear independent vectors $\boldsymbol{\rho}^{k,r},\ r = 1, \ldots, d$. For each $\boldsymbol{\rho} \in \mathcal{B}_d^k$ there exist $\mu_1, \ldots, \mu_d \in \mathbb{R}$ such that:

$$\boldsymbol{\rho} = \mu_1 \boldsymbol{\rho}^{k,1} + \mu_2 \boldsymbol{\rho}^{k,2} + \cdots + \mu_d \boldsymbol{\rho}^{k,d}. \tag{12}$$

Further, the unknown $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_d)^T \in \mathbb{R}^d$ replaces the unknown residual stresses $\boldsymbol{\rho} \in \mathcal{B}_d^k$. The base vectors $\boldsymbol{\rho}^{k,r}, r = 1, \ldots, d$ are assembled into the matrix $\mathbf{B}^{d,k}$:

$$\mathbf{B}_i^{d,k} = \left(\boldsymbol{\rho}_i^{k,1}, \ldots, \boldsymbol{\rho}_i^{k,d} \right).$$

With the stresses $\boldsymbol{\sigma}_i^{0,k-1}$ the perfect plastic problem which has to be solved in the iteration step k is:

$$\begin{aligned} \max \quad & \alpha \\ \text{s.t.} \quad & F[\alpha \boldsymbol{\sigma}_i^E(j) + \boldsymbol{\sigma}_i^{0,k-1} + \mathbf{B}_i^{d,k} \boldsymbol{\mu}] \leqslant \sigma_{y,i}^2 \\ & \text{for } i = 1, \ldots, NG,\ \ j = 1, \ldots, NV,\ \ \boldsymbol{\mu} \in \mathbb{R}^d. \end{aligned} \tag{13}$$

This convex problem has $d + 1$ ($\boldsymbol{\mu}$ and α) unknowns and $NG \cdot NV$ constraints. For obtaining its solution α^k, one can use any optimization algorithm. If the point $(\alpha^k, \boldsymbol{\mu}^k)$ is a feasible point for the k–th step, then

$$\boldsymbol{\rho}_i^{0,k} = \boldsymbol{\rho}_i^{0,k-1} + \mathbf{B}_i^{d,k} \boldsymbol{\mu}^k$$

and the next iteration can be performed. At the beginning of the iterative process $\boldsymbol{\sigma}^{0,0} = \boldsymbol{\sigma}^0$. It is obvious that the choice made for $\boldsymbol{\sigma}_i^{0,k}$ assures that $\alpha^{k-1} \leqslant \alpha^k$ for each iteration. If the relative improvement $(\alpha^k - \alpha^{k-1})/\alpha^{k-1}$ is smaller than a given constant, the algorithm stops.

This reduction technique generalizes the line search technique, well–known in optimization theory (Fletcher, 1987). Instead of searching the whole feasible region for the optimum a subspace with a small dimension is chosen and one searches for the best value in this subspace.

It has to be clarified how the base vectors of the residual subspace $\mathcal{B}_d^k$ have to be chosen.

6. Generation of the residual stresses

The residual stresses are generated in each iteration. At the beginning of the $(k+1)$-

th iteration, the load factor α^k and the actual stresses corresponding to each load vertex j are known

$$\boldsymbol{\sigma}_i^k(j) = \alpha^k \boldsymbol{\sigma}_i^E(j) + \boldsymbol{\sigma}_i^{0,k-1} + \mathbf{B}_i^{d,k} \boldsymbol{\mu}^k, \quad i = 1, \ldots, NG.$$

Let $\mathcal{J}^k$ be the set of load vertices active in step k, i.e.

$$j \in \mathcal{J}^k - \exists\, i \text{ such that } F\left[\boldsymbol{\sigma}_i^k(j)\right] = \sigma_{y,i}^2 .$$

For each j in $\mathcal{J}^k$ the residual stresses are generated during the plastic iteration proposed in PERMAS IV (PERMAS, 1988) for solving an elastoplastic problem. The version PERMAS 7 offers several modern nonlinear solvers and for this reason an implementation of the shakedown analysis in PERMAS 7 is in preparation.

We begin with a load $\mathbf{R}$ for which the first yielding takes place in some points of the structure and the load increment $\Delta\mathbf{R}$ is applied. In the iteration $k+1$, we use $\mathbf{R} = \alpha^k \mathbf{P}(j) + \mathbf{P}_0$ and $\Delta\mathbf{R} = \gamma \mathbf{P}(j)$ where the dead load $\mathbf{P}_0$ is in equilibrium with the stress $\boldsymbol{\sigma}^0$, $j \in \mathcal{J}^k$ and γ a parameter.

The iterative scheme used by PERMAS IV is based on the 'initial strain' method and can be summarized as follows:

1. The actual stress $\boldsymbol{\sigma}$ corresponds to the considered load. Start with the estimated equivalent plastic strain increment $\overline{\Delta\boldsymbol{\varepsilon}^p} = 0$.
2. Calculate the load increment $\Delta\mathbf{Q}_0$ corresponding to $\Delta\boldsymbol{\varepsilon}^p = \overline{\Delta\boldsymbol{\varepsilon}^p}\dfrac{\partial F}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma})$. (For each element $\mathcal{E}$, $\Delta\mathbf{Q}_0 = \int_{\mathcal{E}} B^T(\mathbf{x})\mathbf{E}\Delta\boldsymbol{\varepsilon}^p(\mathbf{x})dV$, with $\mathbf{E}$ the isotropic elasticity matrix and B the strain–displacement matrix.)
3. Solve the discretized elastic equilibrium system $K\Delta\mathbf{q} = \Delta\mathbf{R} + \Delta\mathbf{Q}_0$, where K is the global assembled stiffness matrix, for finding the displacement increment $\Delta\mathbf{q}$. (K is not updated during the iteration scheme.)
4. Calculate the strain increment $\Delta\boldsymbol{\varepsilon}$, the actual stress $\boldsymbol{\sigma}$ and the equivalent plastic strain increment $\overline{\Delta\boldsymbol{\varepsilon}^p}$. For the elastic–perfectly plastic material

$$\overline{\Delta\boldsymbol{\varepsilon}^p} = \begin{cases} \dfrac{1}{\mathbf{s}\,\mathbf{E}\,\mathbf{s}}(F(\boldsymbol{\sigma}) - \sigma_y^2), & \text{if } F(\boldsymbol{\sigma}) \geqslant \sigma_y^2 \\ 0, & \text{otherwise.} \end{cases}$$

   In the previous relation, $\mathbf{s} = \dfrac{\partial F}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma})$ is calculated with the stresses computed in this step.
5. Compare the estimated value of the equivalent plastic strain increment with the calculated value and if the convergence criteria is not fulfilled return to Step 2 with the current actual stress and the calculated equivalent plastic strain increment.

With the notation $\widehat{\boldsymbol{\sigma}}_i = \boldsymbol{\sigma}_i - \mathbf{E}\Delta\boldsymbol{\varepsilon}_i^p$, for each iteration of the described scheme the following relations can be derived from the virtual work principle:

$$\sum_{i=1}^{NG} C_i \cdot \left(\mathbf{E}\Delta\boldsymbol{\varepsilon}_i^1 - \widehat{\boldsymbol{\sigma}}_i^0\right) = \mathbf{R} + \Delta\mathbf{R}, \ldots ,$$
$$\sum_{i=1}^{NG} C_i \cdot \left(\mathbf{E}\Delta\boldsymbol{\varepsilon}_i^{d+1} - \widehat{\boldsymbol{\sigma}}_i^d\right) = \mathbf{R} + \Delta\mathbf{R}.$$

Here, $\Delta\boldsymbol{\varepsilon}_i^r$ and $\widehat{\boldsymbol{\sigma}}_i^r$ are the quantities obtained in the r–th iteration of the scheme. Of course, the parameter κ must be large enough for allowing d iterations of the scheme before the convergence criteria mentioned in Step 5 is achieved. Subtracting the first equation from the others, the stresses

$$\boldsymbol{\rho}_i^m = \mathbf{E}\Delta\boldsymbol{\varepsilon}_i^{m+1} - \widehat{\boldsymbol{\sigma}}_i^m - \mathbf{E}\Delta\boldsymbol{\varepsilon}_i^1 - \widehat{\boldsymbol{\sigma}}_i^0, \text{ with } m = 1, \ldots , d$$

are obviously residual stresses.

7. Proposed method for bounded kinematic hardening

The basis reduction and the subspace iteration technique described in section 5 cannot be directly applied to the shakedown problem for bounded kinematic hardening model. A method using the overlay model and the basis reduction was developed in (Zhang, 1991), (Stein et al., 1993). The overlay model imposes that all the layers are discretized in the same way, i.e. the elements which lay on top of each other have the same nodes. Therefore, the implementation described in these papers can be applied only for two–dimensional finite elements or for particular three–dimensional finite elements. The method proposed in this section is applicable with arbitrary three–dimensional finite elements.

Under the condition

$$\sigma_u < 2\sigma_y \tag{14}$$

we propose a new method for estimating the shakedown load factor corresponding to a bounded kinematic hardening behavior described through the constitutive equations of the first paragraph.

Let α_{pp} be the solution of the optimization problem corresponding to the perfectly plastic case:

$$\begin{aligned} \max \quad & \alpha \\ \text{s.t.} \quad & F[\alpha\boldsymbol{\sigma}_i^E(j) + \boldsymbol{\rho}_i] \leqslant \sigma_{y,i}^2 \\ & \text{for } i = 1, \ldots , NG, \quad j = 1, \ldots , NV, \quad \boldsymbol{\rho} \in \mathcal{B}. \end{aligned} \tag{15}$$

The basis reduction technique presented in a previous section can be used for this problem. In this case the stress $\boldsymbol{\sigma}^0$ is zero. Let $\boldsymbol{\rho}_{pp}$ be a residual stress such that

$(\alpha_{pp}\ , \boldsymbol{\rho}_{pp})$ is a feasible point for the above problem and at least for one Gaussian point i^* and one load vertex j^*, the equality is achieved (i.e. the vertex j^* is active). Corresponding to this load vertex the back–stress $\boldsymbol{\pi}^*$ is chosen:

$$\boldsymbol{\pi}_i^* = \frac{\sigma_{u,i} - \sigma_{y,i}}{\sigma_{y,i}} \left(\alpha_{pp}\boldsymbol{\sigma}_i^E(j^*) + \boldsymbol{\rho}_{pp,i}\right) \text{ with } i = 1, \dots , NG.$$

The following optimization problem gives an estimation of the bounded kinematic hardening load factor (α_{sd}):

$$\begin{aligned} \max \quad & \alpha \\ \text{s.t.} \quad & F[\alpha\boldsymbol{\sigma}_i^E(j) + \boldsymbol{\rho}_i - \boldsymbol{\pi}_i^*] \leqslant \sigma_{y,i}^2 \\ & \text{for } i = 1, \dots , NG, \quad j = 1, \dots , NV, \quad \boldsymbol{\rho} \in \mathcal{B}. \end{aligned} \tag{16}$$

The basis reduction technique applies to the problem (16), this time with the stresses $\boldsymbol{\sigma}^0 = -\boldsymbol{\pi}^*$. The condition (14) assures that $(0, \mathbf{0})$ is a feasible point for this problem, therefore its admissible set is non-empty.

The solution α^* of the problem (16) is an estimation of the load factor α_{sd}.

If $(\alpha, \boldsymbol{\rho})$ is a feasible point for the problem (16), then $(\alpha, \boldsymbol{\rho}, \boldsymbol{\pi}^*)$ is a feasible point for the optimization problem which gives the shakedown load factor α_{sd} for the bounded kinematic hardening behavior i.e. for the problem:

$$\begin{aligned} \max \quad & \alpha \\ \text{s.t.} \quad & F[\alpha\boldsymbol{\sigma}_i^E(j) + \boldsymbol{\rho}_i - \boldsymbol{\pi}_i] \leqslant \sigma_{y,i}^2 \\ & F[\boldsymbol{\pi}_i] \leqslant \left(\sigma_{u,i} - \sigma_{y,i}\right)^2 \\ & \text{for } i = 1, \dots , NG, \quad j = 1, \dots , NV, \quad \boldsymbol{\rho} \in \mathcal{B}, \ \boldsymbol{\pi} \in \mathbb{R}^{NS \cdot NG}. \end{aligned} \tag{17}$$

It follows that $\alpha^* \leqslant \alpha_{sd}$.

Also, we must notice that if $(\alpha, \boldsymbol{\rho})$ is a feasible point for the problem (15), then $\sigma_y/\sigma_u\,(\alpha, \boldsymbol{\rho})$ is a feasible point for the problem (17). Consequently, the greatest possible value of α_{sd} is $\sigma_u/\sigma_y\alpha_{pp}$. The constants σ_y and σ_u denote the minimum, respectively the maximum, over all the Gaussian points $\mathbf{x}_i$ of $\sigma_{y,i}$ and $\sigma_{u,i}$, respectively.

REMARK 1. Let us consider the particular load domain $\mathcal{L} = [\mathbf{0}, \mathbf{P}]$, i.e. $\mathcal{L}$ is the convex set generated by the load vertices $\mathbf{0}$ and $\mathbf{P}$. For homogeneous material the yield and the ultimate stress do not vary with the Gaussian points. In this case, if $(\alpha, \boldsymbol{\rho})$ is a feasible point for the problem (15), then $F[\boldsymbol{\rho}_i] \leqslant \sigma_y^2$ for each i and it follows easily that $\left(\alpha, \left(2 - \sigma_u/\sigma_y\right)\boldsymbol{\rho}\right)$ is a feasible point for the problem (16). Consequently, in this particular case $\alpha_{pp} \leqslant \alpha^*$.

REMARK 2. In limit analysis, i.e. for the load domain $\mathcal{L} = \{\mathbf{P}\}$, if the yield and the ultimate stresses are constant then a well–known result proves that $\alpha_{sd} = \sigma_u/\sigma_y\alpha_{pp}$. Moreover, it follows easily that in this hypotheses also $\alpha^* = \alpha_{sd}$. For a general load domain this assertion is not true anymore, α_{sd} could take any value in the closed interval $\left[\alpha_{pp}, \sigma_u/\sigma_y\alpha_{pp}\right]$.

8. Implementation

For handling a wide range of structures it was decided to use the commercial FEM-Code PERMAS (INTES, Stuttgart). This code infers in the implemented method in two parts. It calculates the fictitious elastic stresses $\boldsymbol{\sigma}^E(j)$ for each load vertex $\mathbf{P}(j)$. Also, it is used for the generation of the residual stresses $\boldsymbol{\rho}^k$ for each iteration k of the reduced basis method. As we have already mentioned, the optimization problems (13) could be treated with any optimization procedure. In our implementation the reduced problem is solved by a self-implemented SQP-method (Sequential Quadratic Programming) with augmented Lagrangian type line search function (Schittkowski, 1981), Armijo's step length rule and BFGS matrix update (Staat and Heitzer, 1997). Due to the small numbers of unknowns and the large number of inequality constraints, the quadratic sub–problems are solved by an active–set–strategy (Fletcher, 1987). Derivatives are calculated analytically avoiding automatic differentiation methods. For more details we refer to (Heitzer, 1999).

The method proposed in the previous paragraph for obtaining an estimation of the shakedown factor has the advantage that instead of solving the optimization problem (10) with $1 + \dim \mathcal{B} + NS \cdot NG$ unknowns, we solve two optimization problems which can be treated with the basis reduction method. Consequently, even for large–scale optimization problems we have to solve a sequence of optimization problems with a small number of unknowns (maximum 7 unknowns).

The numerical tests performed for the mechanical problems described in the next paragraph give values of α^* which are superior to α_{pp}. For particular load domains the new method gives a value of α^* equal to the limit value $\sigma_u/\sigma_y \alpha_{pp}$.

For the considered examples, a reiteration of the method proposed in Section 7 doesn't give an improvement of the load factor α^*. We expect that if the residual stress $\boldsymbol{\rho}^*$ is chosen such that $(\alpha^*, \boldsymbol{\rho}^*)$ is a feasible point for the problem (16) and if the fictitious elastic stress $\widetilde{\boldsymbol{\sigma}}^E$ corresponds to an active load vertex, then α^* is an approximation for the numerical solution of the problem

$$\begin{aligned} \max \quad & \alpha \\ \text{s.t.} \quad & F[\alpha \boldsymbol{\sigma}_i^E(j) + \boldsymbol{\rho}_i - \widetilde{\boldsymbol{\pi}}_i] \leqslant \sigma_{y,i}^2 \\ & \text{for } i = 1, \dots, NG, \quad j = 1, \dots, NV, \quad \boldsymbol{\rho} \in \mathcal{B} \end{aligned}$$

with the back–stress $\widetilde{\boldsymbol{\pi}}$ given by

$$\widetilde{\boldsymbol{\pi}}_i = \frac{\sigma_{u,i} - \sigma_{y,i}}{\sigma_{y,i}} \left(\alpha^* \widetilde{\boldsymbol{\sigma}}_i^E + \boldsymbol{\rho}_i^*\right) \qquad i = 1, \dots, NG.$$

Therefore, we consider that a better estimation of the shakedown load factor α_{sd} cannot be obtained in this way.

The numerical tests have shown that the particular choice of an active load vertex j^* has no influence on the value obtained for α^*. We intend to study in further research, the effect of considering a back–stress comprising the simultaneous influence of several active load vertices.

9. Numerical results

9.1. PROBLEM 1

A thin rectangular plate supported in the corners in the vertical direction is considered. The tension p is applied on the lateral sides and the temperature T is equally distributed on the plate (see Figure 1). The numerical results for the bounded kinematic hardening behavior correspond to the choice $\sigma_u = 1.5\sigma_y$. Due to the symmetry of the problem, only a quarter of the plate is considered. The nodes on the side $x = 0$ can move only in the horizontal direction and the nodes on the side $y = 0$ only in the vertical direction. Because of the symmetry of the problem we have used only one 9-noded quadrilateral plane membrane element QUAM9 (PERMAS, 1988). The load factors corresponding to the elastic, the perfectly plastic and the bounded kinematic hardening behavior were computed for different ratios of p and T.

The load domain $\mathcal{L}$ represented in the space tension–temperature has four load vertices:

$$\mathbf{P}(1) = (p, 0),\ \mathbf{P}(2) = (0, T),\ \mathbf{P}(3) = (p, T),\ \mathbf{P}(4) = (0, 0).$$

The enlarged domain $\alpha\mathcal{L}$ is completely determined by the load vertex $(\alpha p, \alpha T)$. The points $(\alpha p, \alpha T)$, where α is the computed load factor, are represented for different ratios of p and T. The obtained numerical results are shown in the Figure 2. The analytical elastic solution for purely mechanical and purely thermal load,

$$p_0 = \frac{1}{\sqrt{1 - \nu + \nu^2}}\sigma_y \quad \text{and} \quad T_0 = \frac{1}{E\alpha}\sigma_y,$$

respectively, are used for scaling. Here, ν is the Poisson's ratio, E is the Young's modulus for the considered material and α is the coefficient of thermal expansion.

We have observed a small influence of the bounded hardening for predominant thermal loadings. A significant increase of the load factor due to the bounded

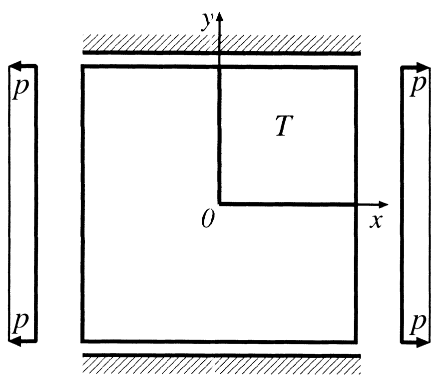

Figure 1. Thin plate.

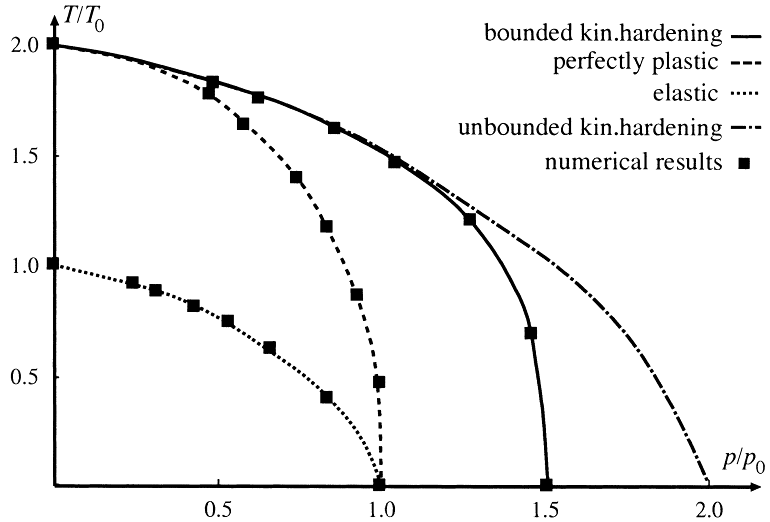

Figure 2. Shakedown diagram for thin plate.

hardening is noticed if the pressure is dominant. The maximal possible shakedown load factor of 1.5 α_{pp} is achieved when there is no temperature load. The curve obtained from the elastic curve through a homothety by factor 2 gives an analytical lower bound of the shakedown load factors for unboounded kinematic hardening behavior. In the purely mechanical loading case the plate yields homogeneously, thus the elastic and perfectly plastic factors coincide. Due to this behavior it is impossible to generate nontrivial residual stresses and therefore numerical problems occur in the optimization algorithm.

9.2. PROBLEM 2

A thin pipe with the radius R and the thickness $d = 0.1\,R$ is fixed in the axial direction. The pressure p and the difference of temperature $\Delta T \geqslant 0$ are applied on the interior side (see Figure 3). The numerical results for the bounded kinematic hardening behavior correspond to the choice $\sigma_u = 1.35\,\sigma_y$. Eight axisymmetric ring elements with quadrilateral cross section QUAX9 (PERMAS, 1988) are used for the discretization. Because of the thinness of the pipe, a linear temperature distribution is chosen. The load factors corresponding to the elastic, the perfectly plastic and the bounded kinematic hardening behavior were computed for different ratios of p and T.

The load domain $\mathcal{L}$ represented in the space pressure–temperature has four load vertices:

$$\mathbf{P}(1) = (p, 0),\ \mathbf{P}(2) = (0, T),\ \mathbf{P}(3) = (p, T),\ \mathbf{P}(4) = (0, 0).$$

The enlarged domain $\alpha\mathcal{L}$ is completely determined by the load vertex $(\alpha p, \alpha T)$. The maximal pressure p_0 computed for purely mechanical loads and the maximal

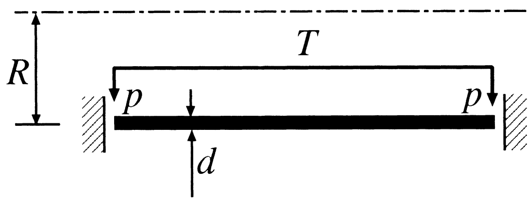

Figure 3. Thin pipe.

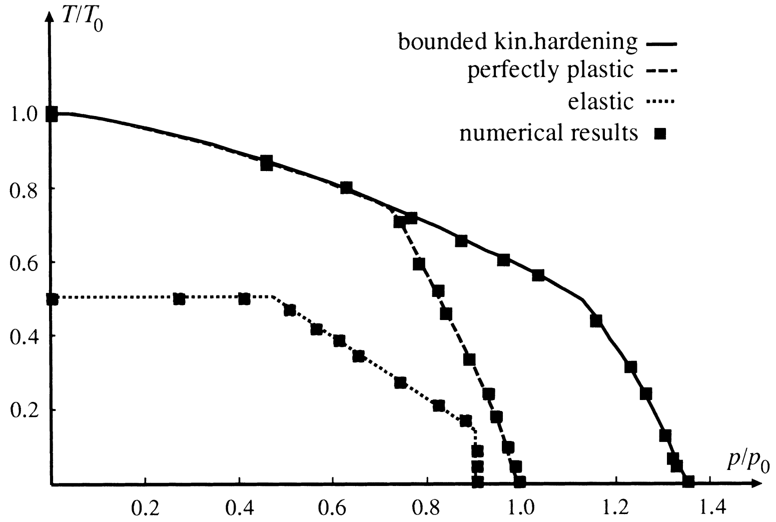

Figure 4. Shakedown diagram for thin pipe.

temperature T_0 for purely thermal loads are used for scaling, both quantities corresponding to the perfectly plastic material behavior with the yield stress σ_y. The points $(\alpha p, \alpha T)$ are represented for different ratios of p and T in Figure 4. No influence of the bounded hardening for predominant thermal loadings is observed. If the temperature loading is not the dominant one, an increase of the load factor due to the bounded hardening is observed. The increase of the load factor due to the considered hardening has been observed for those ratios of p and T for which the influence of the mechanical load on the initial yielding is significant.

Acknowledgements

The research has been partly funded by the European Commission as part of the Brite-EuRam III project LISA: FEM-Based Limit and Shakedown Analysis for Design and Integrity Assessment in European Industry (Project N°: BE 97-4547, Contract N°: BRPR-CT97-0595).

The authors would like to express their gratitude to Prof. P.D. Panagiotopoulos for having planned and organized the LISA project with the last author. Prof.

Panagiotopoulos died seven months after the project start. We will remember him as a friend who made our collaboration possible.

References

Argyris, J. and Mlejnik, H.-P. (1986–1987), *Die Methode der Finiten Elemente: Band I – II*, Braunschweig, Vieweg.

Besseling, J. F. (1985), Models of metal plasticity: theory and experiment, in Sawczuk, A. and Bianchi, G. (eds.): *Plasticity Today*, London, New York, Elsevier Applied Science Publishers, 97–113.

Fletcher, R. (1987), *Practical Methods of Optimization*, New York, John Wiley & Sons.

Gross-Weege, J. (1996), On the numerical assessment of the safety factor of elastic-plastic structures under variable loading, *International Journal of Mechanical Sciences* 39, 417–433.

Halphen, B. and Nguyen, Q. S. (1975), Sur les matériaux standards généralisés, *J. Mécanique* 14, 39–63.

Heitzer, M. (1999), Traglast- und Einspielanalyse zur Bewertung der Sicherheit passiver Komponenten, *Berichte des Forschungszentrums Jülich* Jül-3704.

Kapoor, A. and Johnson, K. L. (1994), Plastic ratchetting as a mechanism of metallic wear, *Proceedings of the Royal Society of London Series A* 445, 367–381.

König, J. A. (1987), *Shakedown of Elastic-Plastic Structures*, Amsterdam and Warschau, Elsevier and PWN.

Melan, E. (1938), Zur Plastizität des räumlichen Kontinuums, *Ingenieur-Archiv* 8, 116–126.

PERMAS (1988), *User's Reference Manuals*, Stuttgart, INTES Publications No. 202, 207, 208, 302, UM 404, UM 405.

Schittkowski, K. (1981), The nonlinear programming method of Wilson, Han and Powell with augmented Lagrangian type line search function, *Numerische Mathematik* 38, 83–114.

Shen, W. (1986), Traglast- und Anpassungsanalyse von Konstruktionen aus elastisch, ideal plastischem Material, Ph.D. thesis, Universität Stuttgart.

Staat, M. and Heitzer, M. (1997), Limit and shakedown analysis for plastic safety of complex structures, *Transactions of SMiRT* 14, B02/2.

Stein, E., Zhang, G., and Mahnken, R. (1993), Shakedown analysis for perfectly plastic and kinematic hardening materials, in Stein, E. (ed.): *Progress in computational analysis of inelastic structures*, Wien, Springer, 175–244.

Weichert, D. and Gross-Weege, J. (1988), The numerical assessment of elastic-plastic sheets under variable mechanical and thermal loads using a simplified two-surface yield condition, *International Journal of Mechanical Science* 30, 757–767.

Zhang, G. (1991), Einspielen und dessen numerische Behandlung von Flächentragwerken aus ideal plastischem bzw. kinematisch verfestigendem Material, Ph.D. thesis, Universität Hannover.